$\mathbb{R}^n$. $\vec{v}, \vec{w}$

$\vec{v}+\vec{w}$, $r\vec{v}$, $\vec{v}-\vec{w}$

dot product $\vec{v}\cdot\vec{w}$. $\|\vec{v}\|^2 = \vec{v}\cdot\vec{v}$

In $\mathbb{R}^2, \mathbb{R}^3$ $\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$

C-S inequality $\|\vec{v}\cdot\vec{w}\| \le \|\vec{v}\|\|\vec{w}\|$

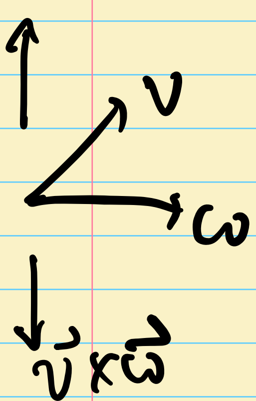
triangle " $\|\vec{v}+\vec{w}\| \le \|\vec{v}\| + \|\vec{w}\|$

cross product ($\mathbb{R}^3$) $\vec{v}\times\vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

$\vec{v}\times\vec{w}$ - orthogonal to $\vec{v}, \vec{w}$

i.e. $(\vec{v}\times\vec{w})\cdot\vec{v} = 0$

( " )$\cdot\vec{w} = 0$.




- right-hand rule

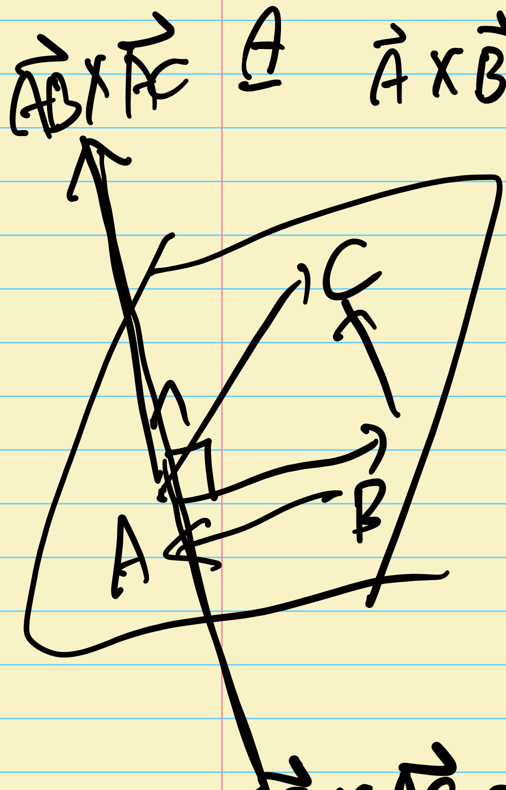
- $\|\vec{v}\times\vec{w}\| = \|\vec{v}\|\|\vec{w}\|\sin\theta$

= area of parallelogram

**Ex** $A=(1,2,1)$, $B=(1,-1,0)$, $C=(2,3,2)$

$ABC$ determines a plane

**Q** Find a normal vector of $P$.




**A** $\vec{A}\times\vec{B}$, $\vec{B}\times\vec{C}$, $\vec{C}\times\vec{A}$ ✗

$\vec{AB}$, $\vec{AC}$

$\vec{AB}\times\vec{AC}$ will do!

$$\vec{AB} = (1,-1,0) - (1,2,1) = (0,-3,-1)$$

$$\vec{AC} = (1,1,1)$$

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -3 & -1 \\ 1 & 1 \end{vmatrix}\hat{i} - \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix}\hat{j} + \begin{vmatrix} 0 & -3 \\ 1 & 1 \end{vmatrix}\hat{k}$$

$$= -2\hat{i} - \hat{j} + 3\hat{k}$$

$\therefore (-2,-1,3)$ is normal to $P$. $\square$

**Rmk** what about $\vec{BA}\times\vec{BC}$? $\vec{AB}\times\vec{BC}$ etc.!

They differ by just scalar multiplication.

**Def** The triple product of $\vec{a}, \vec{b}, \vec{c}$ is
$$\vec{a}\cdot(\vec{b}\times\vec{c})$$

$\vec{a}=(a_1, a_2, a_3)$, $\vec{b}=(b_1, b_2, b_3)$, $\vec{c}=(c_1, c_2, c_3)$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (a_1, a_2, a_3)\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (a_1, a_2, a_3)\cdot\left(\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}, -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}, \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\right)$$

$$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \checkmark .$$

**Prop** $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$

$= -\vec{a}\cdot(\vec{c}\times\vec{b}) = -\vec{b}\cdot(\vec{a}\times\vec{c}) = -\vec{c}\cdot(\vec{b}\times\vec{a})$

(proof) Recall $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

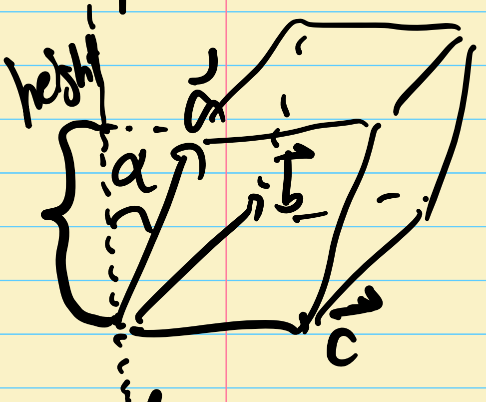
property of determinant.

$$= \vec{b} \cdot (\vec{c} \times \vec{a}).$$

$\square$

geometric meaning

**Prop** $|\vec{a} \cdot (\vec{b} \times \vec{c})|$ = the volume of parallelepiped spanned by $\vec{a}, \vec{b}, \vec{c}$




(proof) For simplicity, suppose $\alpha \leq \frac{\pi}{2}$.

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \|\vec{a}\| \, \|\vec{b} \times \vec{c}\| \cos\alpha$$

$$= \|\vec{b} \times \vec{c}\| \left( \|\vec{a}\| \cos\alpha \right)$$

$$= \|\vec{b} \times \vec{c}\| \cdot (\text{height})$$

$= (\text{area of the base}) \times (\text{height})$

$= \text{volume}.$

If $\frac{\pi}{2} < \alpha \leq \pi$: similar

$\vec{a} \cdot (\vec{b} \times \vec{c}) = -\text{volume}$

$\| \quad \text{"} \quad \| = \text{volume}$ $\square$

<u>Rmk</u> $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

$$\Leftrightarrow \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

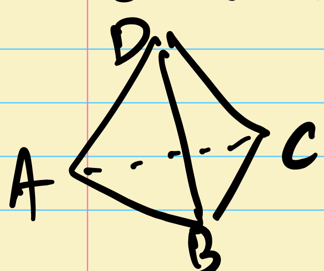
$\Leftrightarrow$ volume $= 0$

$\Leftrightarrow$ $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent.

(linear algebra)

In particular, $A, B, C, D \in \mathbb{R}^3$ 4 points

consider the tetrahedron $ABCD$




volume of $ABCD$

$$= \frac{1}{3}\,(\text{area of } \triangle ABC)\cdot \text{height}$$

$$= \frac{1}{3}\underbrace{\left(\frac{1}{2}\cdot \text{area of parallelogram spanned by } \vec{AB},\ \vec{AC}\right)\cdot \text{height}}_{\parallel}$$

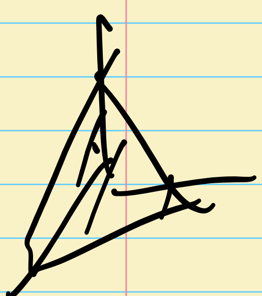
$$= \frac{1}{6}\cdot \text{volume of parallelepiped.}$$

$$= \frac{1}{6}\cdot\left|\vec{AD}\cdot(\vec{AB}\times\vec{AC})\right|$$

Ex $A=(0,0,0)$ $B=(1,0,0)=\hat{i}$

$C=(0,1,0)=\hat{j}$

$D=(0,0,1)=\hat{k}$



$$\text{volume} = \frac{1}{3}\times\left(\frac{1}{2}\times 1\times 1\right)\times 1 = \frac{1}{6}$$

Also, $\frac{1}{6}\left|\vec{AB}\cdot(\vec{AC}\times\vec{AD})\right|$

$$= \frac{1}{6}\left|i\cdot(j\times k)\right|$$

$$= \frac{1}{6}\left|i\cdot i\right| = \frac{1}{6}$$

equal.

Ex $A=(1,0,1)$ $B=(1,1,2)$ $C=(2,1,1)$ $D=(2,1,3)$

volume of ABCD?

(sol) $\overrightarrow{AB}=(1,1,2)-(1,0,1)=(0,1,1)$

$\overrightarrow{AC}=(2,1,1)-(1,0,1)=(1,1,0)$

$\overrightarrow{AD}=(2,1,3)-(1,0,1)=(1,1,2)$

$$\overrightarrow{AD}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})=\begin{vmatrix}1&1&2\\0&1&1\\1&1&0\end{vmatrix}=$$

$$=\begin{vmatrix}1&1\\1&0\end{vmatrix}-\begin{vmatrix}0&1\\1&0\end{vmatrix}+2\begin{vmatrix}0&1\\1&1\end{vmatrix}$$

$$=-1+1-2$$

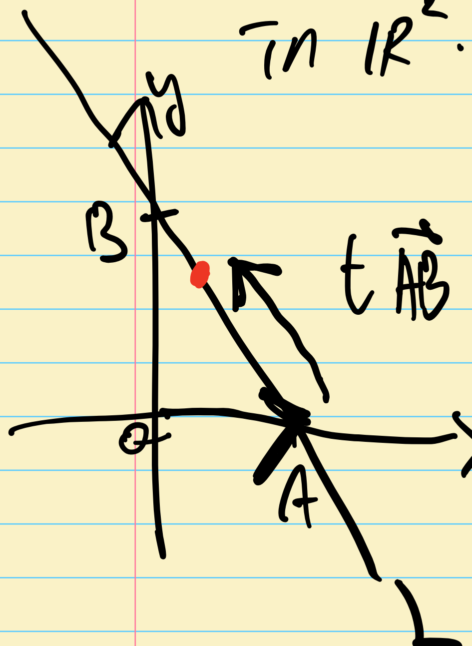
$$=-2.$$

$\therefore$ volume of ABCD $=\frac{1}{6}\times|-2|=\frac{1}{3}.$ $\square$

Geometric objects in $\mathbb{R}^n$ (e.g. line, plane, curve etc.)

Linear objects in $\mathbb{R}^n$ ( line, plane ..)

$L$: a line passing thru $A=(1,0)$, $B=(0,2)$ in $\mathbb{R}^2$.




- equation form

$$2x+y=2$$

- parametric form.

any point on $L$

$$= \vec{OA} + t\vec{AB} \quad (t\in\mathbb{R})$$

$\vec{AB} = (-1,2)$

$$= (1,0) + t(-1,2)$$

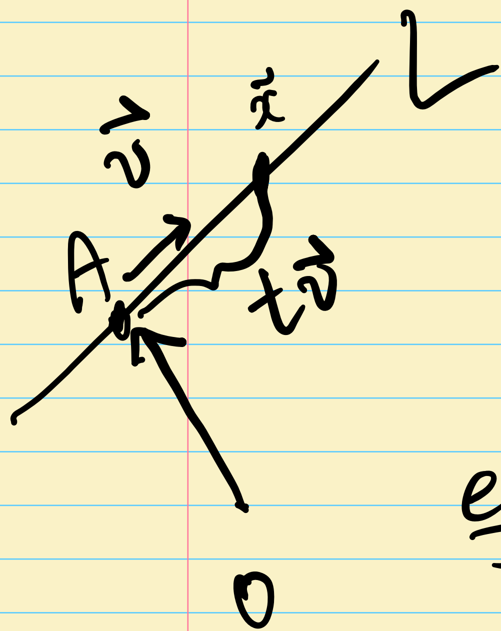
$$= (1-t,\ 2t)$$

- symmetric form

$x=1-t$ $\Rightarrow$ $t=1-x$ $\Rightarrow$ $\dfrac{x-1}{-1} = \dfrac{y-0}{2}$

$y=2t$ $\quad$ $t=\dfrac{y}{2}$

More generally, a line in $\mathbb{R}^n$




$$\vec{x} = \vec{a} + t\vec{v} \quad (t \in \mathbb{R})$$

parametric form

"L is parametrized by $t \in \mathbb{R}$."

eg Find a parametric form of a line L passing thru $A = (1, 2, 3)$ and $B = (-1, 3, 5)$

(sol) we take $\vec{a} = \vec{OA} = (1, 2, 3)$

$$\vec{v} = \vec{AB} = (-1-1,\ 3-2,\ 5-3)$$

$$= (-2, 1, 2)$$

$\therefore$ $(x, y, z)$ on L

$$= (1, 2, 3) + t(-2, 1, 2)$$

$$= (1-2t,\ 2+t,\ 3+2t). \qquad \square$$

**Rmk** · parametric form is not unique

we may take $\vec{a} = (-1, 3, 5) = B$

$\vec{v} = \vec{BA}$ etc.

· parametric form $\Rightarrow$ symmetric form.

eg $(x, y, z) = (1-2t, 2+t, 3+2t)$

$$t = \frac{x-1}{-2} = \frac{y-2}{1} = \frac{z-3}{2}.$$

---

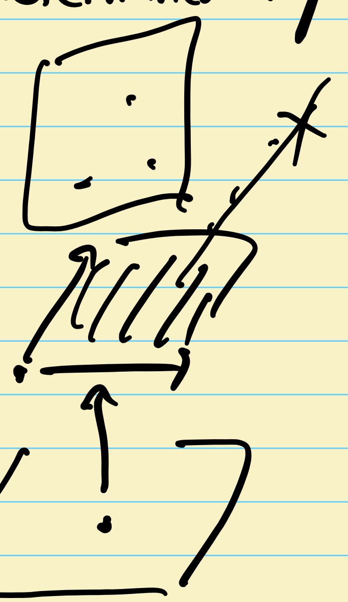
Planes in $\mathbb{R}^3$

A plane $P$ in $\mathbb{R}^3$ can be determined by

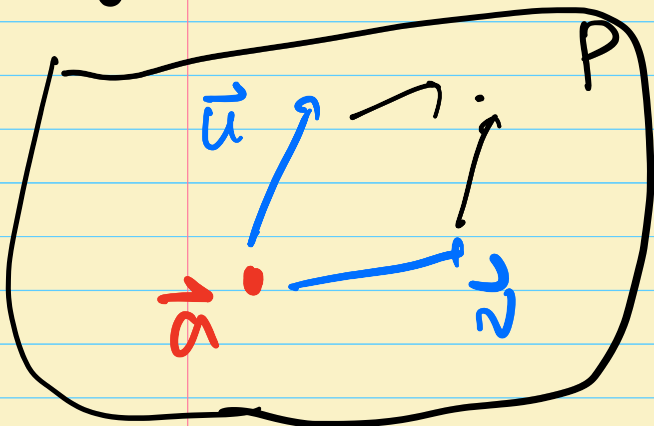
① 3 non-colinear points on $P$

② 1 point & 2 linearly independent vectors

③ 1 point & a normal vector



Using ②; Parametric form of $P$




$\vec{a}$: given point

$\vec{u}, \vec{v}$: given 2 directions

Any point $\vec{x}$ on $P$ is

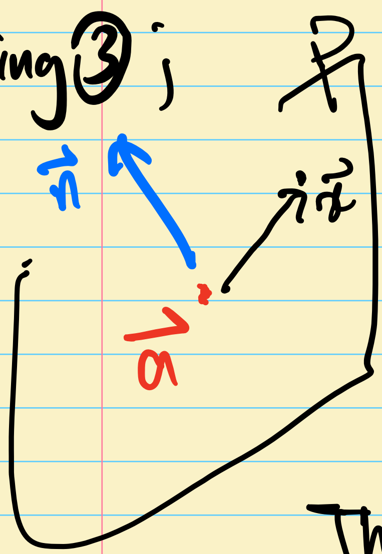
$$\vec{x} = \vec{a} + s\vec{u} + t\vec{v}$$

a point on $P$

directions

$s, t \in \mathbb{R}$ parameters

Using ③;




$\vec{a} = (a_1, a_2, a_3)$ a point

$\vec{n} = (n_1, n_2, n_3)$ a normal vector of $P$.

$\vec{x}$ a point in $\mathbb{R}^3$

Then $\vec{x}$ is on $P \iff \vec{x} - \vec{a} \perp \vec{n}$

$$\iff (\vec{x} - \vec{a}) \cdot \vec{n} = 0$$

$$\iff \vec{x} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

In other words, $\vec{x} = (x_1, x_2, x_3)$

$$\vec{x} \cdot \vec{n} = \vec{x} \cdot \vec{a}$$

$$\Longleftrightarrow \quad n_1 x_1 + n_2 x_2 + n_3 x_3 = \underbrace{n_1 a_1 + n_2 a_2 + n_3 a_3}_{\text{constant.}}$$

**Rmk** $\boxed{ax + by + cz = d}$ $\rightsquigarrow$ a plane in $\mathbb{R}^3$

$\Rightarrow$ $(a, b, c)$ is the normal vector of given plane.

**Example** P: a plane passing thru

$A = (0, 0, 1)$, $B = (0, 2, 0)$, $C = (-1, 1, 0)$

(i) parametric form of P ?

(ii) equation form of P ?

(sol) (i) A point on P ; $A = (0, 0, 1)$

two directions ; $\vec{AB} = (0, 2, 0) - (0, 0, 1)$

$= (0, 2, -1)$

$\vec{AC} = (-1, 1, 0) - (0, 0, 1)$

$= (-1, 1, -1)$

$\therefore$ Parametric form of P:

$$(x, y, z) = (0, 0, 1) + s(0, 2, -1) + t(-1, 1, -1)$$

$s, t \in \mathbb{R}$

(ii) normal vector:

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 0 & 2 & -1 \\ -1 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} i - \begin{vmatrix} 0 & -1 \\ -1 & -1 \end{vmatrix} j + \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} k$$

$$= -i + j + 2k.$$

$$= (-1, 1, 2)$$

$\therefore$ equation form is

$$\left( (x, y, z) - (0, 0, 1) \right) \cdot (-1, 1, 2) = 0$$

$$(x, y, z-1) \cdot (-1, 1, 2) = 0$$

$$-x + y + 2(z-1) = 0 \iff -x + y + 2z = 2$$

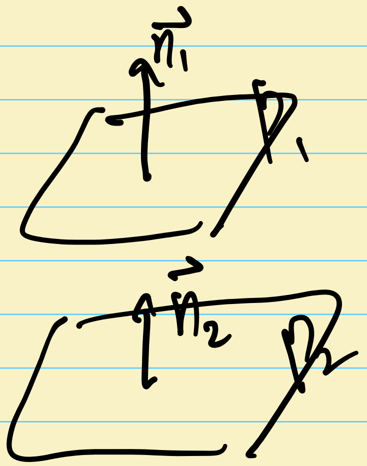
**Example** Two planes in $\mathbb{R}^3$

normal vector

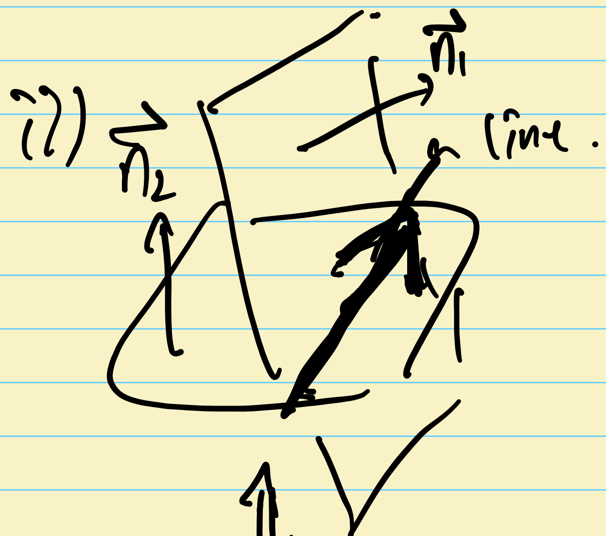
$P_1: a_1x + b_1y + c_1z = d_1$ $\quad \vec{n}_1 = (a_1, b_1, c_1)$

$P_2: a_2x + b_2y + c_2z = d_2$ $\quad \vec{n}_2 = (a_2, b_2, c_2)$

i)




ii)




i) $\Updownarrow$ normal vectors are parallel $\Updownarrow$ $(a_1, b_1, c_1) = r(a_2, b_2, c_2)$

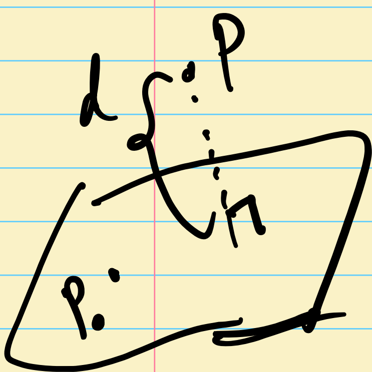
ii) $\Updownarrow$ normal vectors are not parallel $\Updownarrow$ $(a_1, b_1, c_1) \neq r(a_2, b_2, c_2)$

~~Q:~~ In ii), we have a line.

Direction of this line: orthogonal to both $\vec{n}_1$ and $\vec{n}_2$.

a $\vec{n}_1 \times \vec{n}_2$ will do.

**Prop** (distance between plane & point)



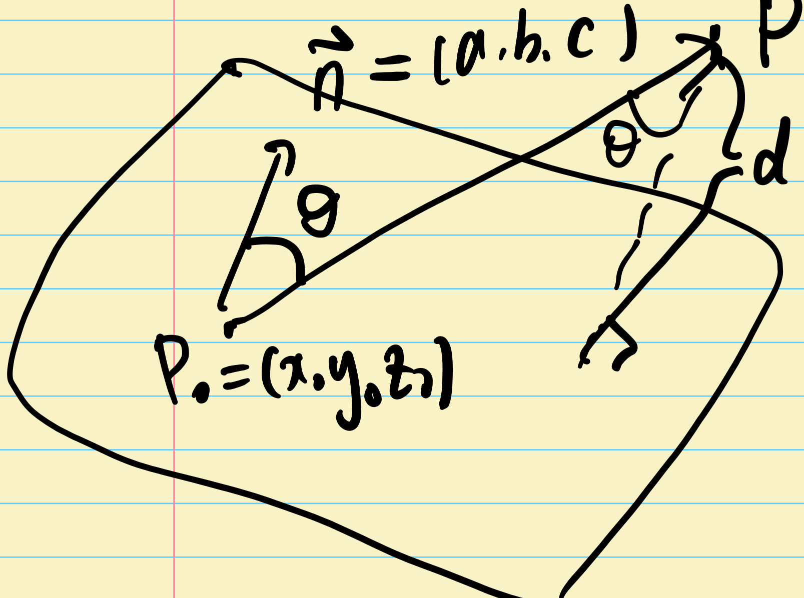

A plane in $\mathbb{R}^3$ is given by

$a(x-x_0)+b(y-y_0)+c(z-z_0)=0.$

$\left(\begin{array}{l} P_0=(x_0,y_0,z_0) \text{ lies on this plane} \\ \vec{n}=(a,b,c) \text{ is a normal vector of} \\ \quad \text{this plane} \end{array}\right)$

A point $P \in \mathbb{R}^3$

The distance $d$ between $P$ and the plane is $d=\left|\overrightarrow{P_0P}\cdot\dfrac{\vec{n}}{\|\vec{n}\|}\right|$




(pf) $\overrightarrow{P_0P}\cdot\vec{n}$

$=\|\overrightarrow{P_0P}\|\,\|\vec{n}\|\cos\theta$

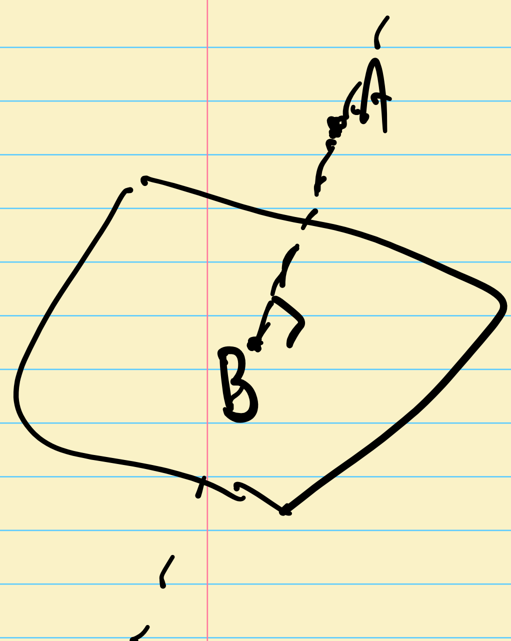
$d=\left|\|\overrightarrow{P_0P}\|\cdot\cos\theta\right|$

$=\left|\dfrac{\overrightarrow{P_0P}\cdot\vec{n}}{\|\vec{n}\|}\right|$

$=\left|\overrightarrow{P_0P}\cdot\dfrac{\vec{n}}{\|\vec{n}\|}\right|$ $\square$

**Example** Find the distance between $A=(2,1,1)$ and a plane $-x+2y-z=-4$.

(sol) From the equation, $\vec{n}=(-1,2,-1)$




We want to find $B$.

$$\vec{B}=\vec{A}+t\vec{n}$$
$$=(2,1,1)+t(-1,2,-1)$$
$$=(2-t,\ 1+2t,\ 1-t)$$

for some $t\in\mathbb{R}$.

$\vec{B}$ lies on the plane;

$$-(2-t)+2(1+2t)-(1-t)=-4$$
$$\Rightarrow 6t-1=-4$$
$$\Rightarrow t=-\frac{1}{2}$$

$\therefore \vec{B}=\left(\frac{5}{2},0,\frac{3}{2}\right)$

$\therefore$ distance $=\|\vec{AB}\|$

$$=\sqrt{\left(\frac{5}{2}-2\right)^2+(0-1)^2+\left(\frac{3}{2}-1\right)^2}$$

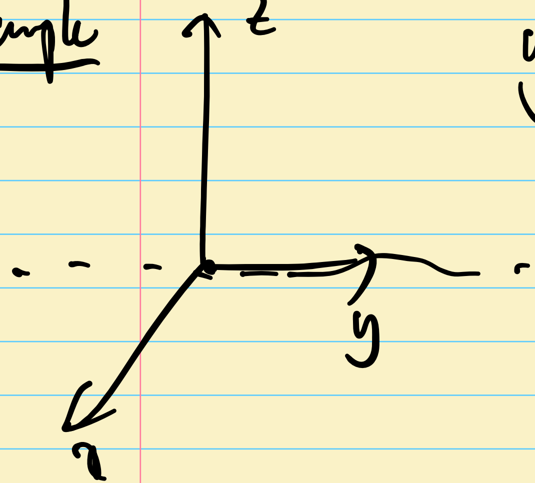
$$=\sqrt{6}/2.$$

$\left[\begin{array}{l}\text{parametric form} \\ \text{equation form.}\end{array}\right.$

plane in $\mathbb{R}^3 \Leftrightarrow 1$ equation.

line in $\mathbb{R}^3$ = plane $\cap$ plane $\Leftrightarrow$ 2 equations.

Example




y-axis in $\mathbb{R}^3$ (a line)

= intersection of xy-plane & yz-plane

= intersection of $\{z=0\}$ & $\{x=0\}$

= points satisfying $\begin{cases} x=0 \\ z=0. \end{cases}$

Example. Let L be the line given by two equations

$$\begin{cases} x+y+6z=6 \\ x-y-2z=-2 \end{cases}$$

Parametric form?

add

$2x+4z=4$

$\Rightarrow x+2z=2.$

$\Rightarrow x=-2z+2.$

$\Rightarrow \cancel{y} \cancel{x =} \; (-2z+2) + y + 6z = 6$

$\Rightarrow y = -4z + 4$

$\therefore$ If $z = t$

$$(x, y, z) \text{ on } L = \underbrace{(-2t+2,\ -4t+4,\ t)}$$

$$= (2, 4, 0) + t(-2, -4, 1)$$

Conversely, given parametric form

$$\left.\begin{array}{l} x = -2t+2 \\ y = -4t+4 \end{array}\right\} \Rightarrow 2x - y = 0$$

$$\left. z = t \quad \right\} \Rightarrow y = 4 - 4z \quad \text{i.e. } y + 4z = 4.$$

$\therefore$ L is an intersection of planes

$$\begin{cases} 2x - y = 0 \\ y + 4z = 4. \end{cases}$$

$$\begin{tikzcd} \text{Equation form} \arrow[r, bend left, "\text{Gaussian elimination}"] & \text{parametric form} \arrow[l, bend left, "\text{eliminating parameters}"] \end{tikzcd}$$